§5.5 Multiple Eigenvalue Sol'ns

previously, saw that, if $n \times n$ matrix, A, has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with associated EV $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ then gen sol'n format to system $\vec{x}' = A\vec{x}$ is:

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \ldots + c_n\vec{v}_n e^{\lambda_n t}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants

next, will explore when char eqn: $\det(A - \lambda I_n) = 0$ does not yield n distinct roots

i.e. eigenvalue has a multiplicity of $k > 1$

multiplicity

may have $< k$ linearly independent associated EV

then it may not be possible to find a "complete" set of linearly independent EV

would say an eigenvalue of multiplicity k is complete if it has k linearly independent associated EV

ex. find gen. sol'n of system $\vec{x}' = \underbrace{\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}}_{A} \vec{x}$

char eqn: $\det(A - \lambda I_n) = 0$

$$\det\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\det\begin{bmatrix} 9-\lambda & 4 \\ -6 & -1-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)[(9-\lambda)(-1-\lambda) + 24] = 0$$

$$(3-\lambda)(-8\lambda + \lambda^2 - 9 + 24] = 0$$

$$(3-\lambda)(\lambda^2 - 8\lambda + 15) = 0$$

$$(3-\lambda)(-8\lambda+\lambda^2-9+24)=0$$

$$(3-\lambda)(\lambda^2-8\lambda+15)=0$$

factor 2 negatives $(3-\lambda)\,(\lambda-3)(\lambda-5)=0$

$$\underbrace{(3-\lambda)\,(3-\lambda)}_{\lambda_1=3}\,\underbrace{(5-\lambda)}_{\lambda_2=5}=0$$

mult: $k=2$ mult: $k=1$

when $\lambda=5$ $\begin{bmatrix}4 & 4 & 0\\ -6 & -6 & 0\\ 6 & 4 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\leftarrow (A-5I_3)\vec{v}=\vec{0}$

I: $4v_1+4v_2=0 \Rightarrow v_1=-v_2$

III: $-6v_2+4v_2-2v_3=0$ $v_2=v_2$

$-2v_2=2v_3 \Rightarrow -v_2=v_3$

$EV_{\lambda=5}=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$

when $\lambda=3$ $\begin{bmatrix}6 & 4 & 0\\ -6 & -4 & 0\\ 6 & 4 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

I: $6v_1+4v_2=0$

$v_2=-\frac{3}{2}v_1$

v_3 is arbitrary

if choose $v_3=1$ $\leftarrow$ non zero value

then $v_1=v_2=0$

$EV_{\lambda=3}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$2\begin{bmatrix}1\\-\frac{3}{2}\\0\end{bmatrix}=\begin{bmatrix}2\\-3\\0\end{bmatrix}$$

if choose $v_3=0$

then v_1 cannot be zero

$EV_{\lambda=3}=\begin{bmatrix}2\\-3\\0\end{bmatrix}$

have a <u>complete</u> set of EV associated with $\lambda=5,3,3$

$$\therefore \vec{x}(t)=c_1\begin{bmatrix}-1\\1\\-1\end{bmatrix}e^{5t}+c_2\begin{bmatrix}0\\0\\1\end{bmatrix}e^{3t}+c_3\begin{bmatrix}2\\-3\\0\end{bmatrix}e^{3t}$$

ex. given $A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$ $\quad \det \begin{bmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix} = 0$

$$(1-\lambda)(7-\lambda) + 9 = 0$$
$$\lambda^2 - 8\lambda + 7 + 9 = 0$$
$$\lambda^2 - 8\lambda + 16 = 0$$
$$(\lambda - 4)^2 = 0$$
$$\lambda = 4 \qquad k = 2$$

then $\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + v_2 = 0 \Rightarrow \begin{matrix} v_1 = v_1 \\ v_2 = -v_1 \end{matrix} \qquad EV_{\lambda=4} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

since a 2nd linearly independent EV does not readily exist, consider $\lambda = 4$ with a multiplicity $k = 2$ to be defective

rule: an eigenvalue λ with multiplicity $k > 1$ is defective if not complete

if λ has $p < k$ linearly independent EV

then $d = k - p$ represents the number of missing EVs

called the defect of λ

in prev. ex., defective $\lambda = 4$ has mult: $k = 2$

assoc EV: $p = 1$

defect: $d = 2 - 1 = 1$

multiplicity

explore when $k = 2$

again consider previous example, when there exists only 1 EV associated with defective λ

so far, only solution of $\vec{x}\,' = A\vec{x}$ is $\vec{x}_1(t) = \vec{v}_1 e^{\lambda t}$

try 2nd soln in format $\vec{x}_2(t) = \vec{v}_1 t e^{\lambda t} + \vec{v}_2 e^{\lambda t}$

where $\vec{v}_1, \vec{v}_2$ are non-zero constant vectors try 2 soln in format $\vec{x}_2(t) = \vec{v}_1 t e^{\lambda t} + \vec{v}_2 e^{\lambda t}$

where $\vec{v}_1, \vec{v}_2$ are non-zero constant vectors

$$\underline{\vec{v}_1 e^{\lambda t}} + \underline{\vec{v}_1 \lambda t e^{\lambda t}} + \underline{\lambda \vec{v}_2 e^{\lambda t}} = \underline{A\vec{v}_1 t e^{\lambda t}} + \underline{A\vec{v}_2 e^{\lambda t}}$$

$$\lambda \vec{v}_1 = A\vec{v}_1 \Rightarrow A\vec{v}_1 - \lambda\vec{v}_1 = \vec{0}$$

$$(A - \lambda I_n)\vec{v}_1 = \vec{0} \quad \text{I}$$

$$\vec{v}_1 + \lambda\vec{v}_2 = A\vec{v}_2 \Rightarrow A\vec{v}_2 - \lambda\vec{v}_2 = \vec{v}_1$$

$$(A - \lambda I_n)\vec{v}_2 = \vec{v}_1 \quad \text{II}$$

I: $(A-\lambda I_n)\vec{v}_1 = \vec{0}$ already in char. eqn format

confirms that $\vec{v}_1$ is EV of A assoc. w λ

plug II into I: $\underbrace{(A-\lambda I_n)\underbrace{(A-\lambda I_n)\vec{v}_2}_{\vec{v}_1}} = \vec{0}$ shows $\vec{v}_2$ CBW in terms of $\vec{v}_1$ in format of char. eqn.

$$\boxed{(A-\lambda I_n)^2 \vec{v}_2 = \vec{0}}$$

will produce 2 linearly independent sol'ns based on defective λ w $k=2$

ex. find gen soln of system $\vec{x}' = \underbrace{\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}}_{A \text{ from previous ex.}} \vec{x}$

know $\lambda = 4$ mult: $k=2$

defective

$(A - 4I_2)(A - 4I_2)$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

matrix multiplication

$$\underset{2\times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} u & w \\ v & z \end{bmatrix}}$$

$$\begin{bmatrix} au+bv & aw+bz \\ cu+dv & cw+dz \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix}$$

$$(A-\lambda I_n)^2 \vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v}_2 = \vec{0}$$

implies that $\vec{v}_2$ can be anything

choose $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then $(A - \lambda I_2)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \vec{v}_1$$

$2 \times 2 \quad 2 \times 1$

solutions: $\vec{x}_1(t) = \vec{v}_1 e^{4t} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t}$

$$\vec{x}_2(t) = \vec{v}_1 t e^{4t} + \vec{v}_2 e^{4t}$$

$$= \begin{bmatrix} -3 \\ 3 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t}$$

$$\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t}$$

$$\begin{bmatrix} -3t + 1 \\ 3t \end{bmatrix} e^{4t}$$

then gen sol'n:

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$= c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} e^{4t}$$

system whose eqns are

$$x_1(t) = -3c_1 e^{4t} - 3c_2 t e^{4t} + c_2 e^{4t}$$

$$x_2(t) = 3c_1 e^{4t} + 3c_2 t e^{4t}$$

factor e^{4t}

$$\boxed{\begin{aligned} x_1(t) &= (-3c_1 - 3c_2 t + c_2)\, e^{4t} \\ x_2(t) &= (3c_1 + 3c_2 t)\, e^{4t} \end{aligned}}$$